NOTATION

T, design temperature; f, input temperature; τ, time; $\Delta\tau$, time step; ω, circular frequency; $\tilde{\omega}$, dimensionless circular frequency; k, fluctuation amplitude attenuation coefficient: q, heat-flux density on the body boundary; a, thermal diffusivity factor; λ, heat-conduction coefficient; h, distance from the body surface to the point of temperature measurement; G, Green's function; J, functional; J_q', gradient of the functional; S_n, direction of descent in the n-th iteration, and η, viscosity index of the iteration algorithm.

LITERATURE CITED

1. O. M. Alifanov, Identification of Heat Transfer Processes of Flying Vehicles [in Russian], Moscow (1979).
2. A. N. Tikhonov and V. Ya. Arsenin, Methods for Solving Incorrect Problems [in Russian], Moscow (1979).
3. E. A. Artyukhin, "Methods and facilities for machine diagnostics of gas turbine engines and their elements," Abstracts of Reports, 2, 58-59, Kharkov (1983).
4. A. V. Lykov, Theory of Heat Conduction [in Russian], Moscow (1967).
5. A. G. Butkovskii, Characteristics of Systems with Distributed Parameters [in Russian], Moscow (1979).
6. I. N. Bronshtein and K. A. Semendyaev, Handbook on Mathematics [in Russian], Moscow (1980).

REGULARIZATION OF THE SOLUTION OF THE INVERSE HEAT-CONDUCTION PROBLEM IN A VARIATIONAL FORMULATION

A. D. Markin and G. G. Pyatyshkin

UDC 536.2

A version of the solution allowing numerical minimization of the target functional to be eliminated is considered.

The effectiveness of a combination of analytical and numerical methods of solution for the analysis of inverse heat-conduction problems (IHP) is a result of many factors. One of the most significant is analytical analysis, which largely determines the algorithm for solution of the IHP as a whole. In this respect, the example of using gradient methods to solve IHP is illustrative [1]. Finding the analytical expression for the gradient of the functional which eliminates the operation of numerical differentiation markedly expands the region of application of the algorithm developed. At present, there is an extensive bibliography on IHP solution; see [2], for example. However, despite the wealth of literature sources, the development of effective and simple computational algorithms even for one-dimensional IHP remains an urgent problem. This is associated with the multiplicity of IHP formulations sometimes requiring separate approaches, the increase in the demands on the accuracy of the results obtained, the appearance of new computational techniques permitting modeling at a qualitatively new level, and so on.

Now consider a version of the regularization of IHP solution in a variational formulation, which combined numerical and analytical methods of analysis and allows a sufficiently simple algorithm for boundary-condition identification to be obtained.

The basis of the approach is to establish the relation between the conditions defining the IHP and the desired boundary conditions [3]. For example, insolving IHP for a plane wall, this relates the known temperature and its gradient at one boundary to the desired temperature or its gradient at the other boundary.

In general form, the variational formulation of IHP was given in [1].

Defining the target functional analogously gives

Donetsk Polytechnic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 52, No. 6, pp. 991-995, June, 1987. Original article submitted April 15, 1986.

0022-0841/87/5206-0717 $12.50

$$J = \int_0^{\tau_m} (t - t^*)^2 \, d\tau, \quad (1)$$

where $t^* = t^*(\tau)$ is the known temperature inside the body (possibly the result of measurement); $t = t(\tau)$ is the theoretical temperature at the measurement point in accordance with the mathematical model of the given process.

Consideration is confined to a Cartesian coordinate system, since the structure of the approximating system of algebraic equations with a tridiagonal coefficient matrix is significant for the future analysis. The mathematical model of a one-dimensional linear case takes the form

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2}, \; \tau \geqslant 0, \; 0 \leqslant x \leqslant b, \; t(b, \tau) = f_1(\tau); \quad (2)$$

$$-\lambda \frac{\partial t}{\partial x}(b, \tau) = f_2(\tau); \; t(x, 0) = f_3(x); \; q_f(\tau) = -\lambda \frac{\partial t}{\partial x}(0, \tau).$$

Numerical minimization of the functional in Eq. (1) taking account of Eq. (2) is possible. However, this version of the search for a solution is associated, in some form, with numerical determination of the gradient of the functional in Eq. (1). Therefore, the system of algebraic equations approximating Eq. (2) is investigated in more detail. For the sake of simplicity, consider a grid with a uniform step denoting the number of the point with respect to τ by i and the number with respect to x by j ($i = 0, 1, 2, \ldots, M$; $j = 0, 1, 2, \ldots, N$).

A strictly implicit system is chosen for the approximation of Eq. (2).

The system of algebraic equations approximating Eq. (2) takes the form

$$At_{j+1}^{i+1} + Bt_j^{i+1} + Ct_{j-1}^{i+1} + D_j^i = 0, \; j = 1, 2, 3, \ldots, N-1,$$

where for the chosen model

$$A = C = \frac{ah_\tau}{h_x^2} = F; \; B = -2F - 1; \; D_j^i = t_j^i. \quad (3)$$

The coefficient matrix of Eq. (3) is tridiagonal in form. The most effective method of solution of this type is the factorization method [4].

The desired solution of Eq. (3) is written in the form

$$t_{j-1}^{i+1} = \alpha_{j-1} t_j^{i+1} + \beta_{j-1} \quad (4)$$

according to the factorization method, and the following theoretical expressions are obtained for the fitting factors

$$\alpha_j = \frac{A}{-B - C\alpha_{j-1}}; \; \beta_j = \frac{C\beta_{j-1} + D_j^i}{-B - C\alpha_{j-1}}. \quad (5)$$

In approximating boundary conditions of the second kind using the expression recommended in [2], the result obtained for the left-hand boundary is

$$\alpha_0 = \frac{F}{1/2 + F}; \; \beta_0 = \frac{q_f F \frac{h_x}{\lambda} + \frac{1}{2} t_0^i}{1/2 + F} \quad (6)$$

and for the right-hand boundary ($f_2 > 0$ on heating)

$$t_N^{i+1} = \frac{f_2 F \frac{h_x}{\lambda} + F\beta_{N-1} + \frac{1}{2} t_N^i}{1/2 + F - \alpha_{N-1} F}. \quad (7)$$

In this case, the discrepancy between the known f and theoretical t_N temperatures is determined as

$$\frac{f_2 F \frac{h_x}{\lambda} + F\beta_{N-1} + \frac{1}{2} t_N^i}{1/2 + F - \alpha_{N-1} F} - f_1^{i+1} = \delta, \tag{8}$$

where δ is the error in specifying the experimental information

Eliminating the quantities $\beta_1, \beta_2, \ldots, \beta_{j-1}, \ldots, \beta_{N-2}$ by means of the recurrence relations in Eq. (5), it is found that

$$\beta_{N-1} = \beta_0 \prod_{m=1}^{N-1} \alpha_m + \sum_{k=1}^{N-1} \left(\frac{D_k}{C} \prod_{m=k}^{N-1} \alpha_m \right). \tag{9}$$

Writing β_{N-1} in the form

$$\beta_{N-1} = q_f F_1 + F_2 \tag{10}$$

and taking account of Eq. (6), the following expressions are obtained for F_1 and F_2

$$F_1 = \frac{F \frac{h_x}{\lambda}}{1/2 + F} \prod_{m=1}^{N-1} \alpha_m;$$

$$F_2 = \frac{t_0}{1/2 + F} \prod_{m=1}^{N-1} \alpha_m + \sum_{k=1}^{N-1} \left(\frac{D_k}{C} \prod_{m=k}^{N-1} \alpha_m \right). \tag{11}$$

Using Eqs. (8) and (10), it is possible to determine the desired functional relation

$$q_f^{i+1}(\tau) = \frac{(\delta + f_1^{i+1}) \left(\frac{1}{2} + F - \alpha_{N-1} F \right) - f_2^{i+1} \frac{h_x}{\lambda} F - F_2 - t_N^i \frac{1}{2}}{F F_1} \tag{12}$$

between the boundary conditions f_1 and f_2 and the arbitrary thermal load q_f. Qualitative confirmation of this conclusion was obtained in [5].

In the subsequent analysis, it is expedient to use the heat-conduction equation in operator form

$$A q_f = b_\delta, \tag{13}$$

where b_δ are the characteristics of the thermal model, measured with an error; A is a linear continuous operator.

It is required that the quadratic deviation of the left-hand side of Eq. (13) from the right-hand side over the whole range of variation of τ be finite (the constraint is ultimately determined by the error in specifying the initial data)

$$\int_0^{\tau_m} (A_{q_f} - b_\delta)^2 d\tau \leqslant \rho^2. \tag{14}$$

Tikhonov has shown that, imposing certain constraints on the casee of permissible functions, in which the solution is sought, the problem may be converted from an incorrect to a conditionally correct formulation. In many practical IHP, it is sufficient to impose constraints on the first derivative of the desired function [6]. This entails minimizing the functional

$$\min_{q_f} I = \min_{q_f} \int_0^{\tau_m} q_f'^2 d\tau, \quad q_f' = \partial q_f / \partial \tau. \tag{15}$$

The set of Eqs. (14) and (15) is regarded as a problem at a conditional extremum. Using the Lagrangian-multiplier method results in the function

TABLE 1. Initial Experimental Data, Theoretical Heat Flux, Theoretical and Accurate Surface Temperature According to the Data of [5] and the Present Method

$\mathrm{Fo}=\frac{a\tau}{b^2}$	t ($x=0,5$)	t, accurate ($x=0$)	numerical	t (present work) numerical	q [5]	q (present work)
			($x=0$)			
0,08	0,040	0,31915	0,22570	0,256	0,80891	0,9060
0,16	0,118	0,45147	0,45873	0,441	1,22240	1,1000
0,24	0,198	0,55436	0,53962	0,537	0,97454	0,9880
0,32	0,278	0,64472	0,63475	0,634	1,00030	1,0001
0,40	0,358	0,72942	0,72280	0,722	1,00010	1,0000
0,48	0,438	0,81156	0,80727	0,807	1,00000	1,0000
0,56	0,518	0,89253	0,88979	0,890	1,00000	1,0000

$$\Phi \equiv q_f'^2 + \mu (A q_f - b_\delta)^2, \tag{16}$$

which must satisfy the Euler equation

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \Phi}{\partial q_f'}\right) - \frac{\partial \Phi}{\partial q_f} = 0. \tag{17}$$

Solving Eqs. (14) and (17), taking account of Eq. (16), offers the possibility of determining the values of q_f and μ corresponding to the regularized solution of the initial problem. Taking account of Eqs. (8) and (10), Eq. (16) takes the form

$$\Phi = q_f'^2 + \mu \left\{ \frac{f_2 F \frac{h_x}{\lambda} + F(q_f F + F_2) + \frac{1}{2} t_N^i}{1/2 + F - \alpha_{N-1} F} - f_1^{i+1} \right\}^2, \tag{18}$$

which leads, by means of the Euler equation, to an ordinary differential equation in terms of the desired function $q_f(\tau)$:

$$\gamma q_f'' - \varphi_1 q_f - \varphi_2 = 0, \tag{19}$$

where $\gamma = 1/\mu$ is the regularization parameter defined by Eq. (14) and

$$\varphi_1 = \frac{(FF_1)^2}{K^2}; \quad \varphi_2 = \frac{f_2 F \frac{h_x}{\lambda} + FF_2 + \frac{t_N^i}{2} - f_1^{i+1} K}{K^2} FF_1;$$

$$K = 1/2 + F - \gamma_{N-1} F.$$

Analysis of Eq. (19) shows that the constraint on the class of permissible functions of tye type in Eq. (15) actually corresponds to the formulation of an additional boundary condition; in fact, Eq. (19) is of second order.

The accuracy of the method proposed for IHP solution is estimated by comparison with the data of [5], for the example of the following methodological problem given in [5]. For a plane wall of unit thickness, which is heat insulated at one surface, the boundary conditions must be established from the known temperature in the middle plane (x = 0.5). The known temperature is determined by accurate solution for constant (unit) thermal load. The grid parameters correspond to those in [5]: $h_x = 0.1$; $h_\tau = 0.08$; $a = 1$.

The results of solving the model problem are shown in Table 1.

Analysis of the numerical investigation of the algorithm realizing this method of identifying the boundary conditions reveals high accuracy and simplicity of computer solution. In particular, converting the solution program for the direct problem to the solution of IHP by the given method requires 22 operators in FORTRAN.

NOTATION

t, temperature; x, τ, spatial and time coordinates; λ, a, thermal conductivity and thermal diffusivity; q, heat flux density, h_x, h_τ, space–time parameters of grid.

LITERATURE CITED

1. O. M. Alifanov, Identification of Airplane Heat-Transfer Processes [in Russian], Moscow (1979).
2. L. A. Kozdoba and P. G. Krukovskii, Methods of Solving Inverse Heat-Transfer Problems [in Russian], Kiev (1979).
3. A. Markin, in: Tepelna Technika a Automatizace Hutnickych Peci, Třinec (1977), pp. 186-196.
4. A. A. Samarskii, Theory of Difference Schemes [in Russian], Moscow (1977).
5. B. Blackwell, Num. Heat Trans., No. 4, 229-238 (1981).
6. A. N. Tikhonov, Dokl. Akad. Nauk SSSR, 151, No. 3, 501-504 (1963).

SINGLE-PHASE PROBLEMS OF THE MELTING OF SOLID WEDGES

A. D. Chernyshov and O. P. Reztsov UDC 536.42

Accurate solutions obtained in quasisteady formulation take the form of finite sums and are valid from a plane two-face aperture angle of kπ, where k is any simple fraction.

The present work is a continuation of [1] and uses the same notation and formulation of the problem of the melting of solids.

1. In Cartesian coordinates (y, z), the quasisteady heat-conduction equation takes the form [1]

$$\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}+\frac{V_0}{a}\frac{\partial U}{\partial z}=0. \tag{1}$$

The boundary conditions of the problem are

$$U(y,\ z)|_\Sigma=0,\quad U(y,\ z)\leqslant 0,\quad r_m\in\Omega, \tag{2}$$

$$U(y,\ z)\to U_\infty\leqslant 0 \quad \text{as} \quad r_m\to\infty, \tag{3}$$

where Ω is the region of melted solid wedge. Dimensionless variables are introduced

$$\xi_j=(z\sin\Theta_j-y\cos\Theta_j)V_0/a,\quad j=1,\ 2. \tag{4}$$

The angles Θ_1 and Θ_2 are measured counterclockwise from the positive direction of the axis to a straight line passing through the corresponding face of the plane wedge (Fig. 1). The equations of the faces of the melting plane wedge here are: $\xi_1 = 0$, $\xi_2 = 0$; for the region inside the wedge, $\xi_1 > 0$, $\xi_2 > 0$. The straight line ξ_1 = const and ξ_2 = const are parallel to the corresponding planes of the wedge. Note that in geometric terms ξ_1 and ξ_2 are the distances from the point with coordinates (y, z) to the corresponding face of the wedge, multiplied by V_0/a. With this definition of Θ_1 and Θ_2, the aperture angle of the wedge φ_0 corresponds to the expression

$$\Theta_2-\Theta_1=\psi_0=\pi-\varphi_0. \tag{5}$$

The heat-conduction equation (1) is now written in new variables

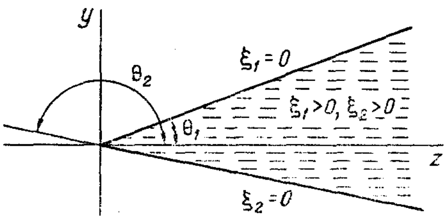

Fig. 1. Cross section of the melting wedge (shaded region).

Vinnitskii Polytechnic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 52, No. 6, pp. 995-999, June, 1987. Original article submitted January 2, 1986.

0022-0841/87/5206-0721 $12.50